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Multicoloured Temperley–Lieb lattice models. The ground state

A Babichenko[1,2] and D Gepner[1]

[1] Department of Particle Physics, Weizmann Institute of Science, Rehovot 76100, Israel
[2] Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

E-mail: babichen@wicc.weizmann.ac.il

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Abstract
Using the inversion relation method, we calculate the ground-state energy for the lattice integrable models, based on baxterization of multicoloured generalization of Temperley–Lieb algebras. The simplest vertex model is analysed and found to have a mass gap.

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1. Introduction

Looking for new solutions of the Yang–Baxter equation is an interesting and important problem in exactly solvable statistical models. Some time ago a set of new solutions of Yang–Baxter equations were obtained [1] by 'baxterization' of multicoloured generalization of Temperley–Lieb (TL) algebras—Fuss–Catalan algebras—discovered by Bisch and Jones [2]. These solutions are really new and are not equivalent to a fusion of known models. In [1], these solutions were formulated in terms of generators of this new algebra, and in principle the Boltzmann weights may be interpreted both as a vertex model or as an interaction round the face (IRF) model defined on a tensor product of admissibility graphs corresponding to different colours of the TL algebra. Physical interpretation of these new integrable models may be different. In addition to construction of new integrable vertex and IRF models, an obvious possible realization of these new solutions is multicoloured generalization of dense loop models [1], polymers and other subjects of exactly solvable statistical models (for a review see, e.g., [3]). Another new integrable model of the same class was obtained as an integrable solution for Lorentz lattice gases [4, 5] and dimerized coupled spin-$1/2$ chain [6].

In spite of the traditional form of the R-matrix, it turns out that the usual algebraic Bethe ansatz solution, at least in its naive and straightforward version, reveals some technical problems. One of the possible reasons for that might be a non-trivial structure of pseudo vacuum, which is hard to guess in a way effective for realization of algebraic Bethe ansatz program. The alternative method for investigation of physical properties of integrable models, such as ground-state energy, its free energy, spectrum and correlation functions is the method of inversion relations. It does not require an explicit form of BA equations or transfer matrix eigenvalues, valid in the thermodynamic limit. It was taken up by Baxter (see, e.g., [7, 8]) and further developed by others (see, e.g., [9, 10]). It exploits functional inversion relations for the transfer matrix in the ground state, and transfer matrix analytical properties, in order to calculate ground-state energy and low-lying excitations' spectrum [10]. Assumptions on the transfer matrix eigenvalue analytical properties should usually be confirmed numerically. The advantage of this method is that it does not require knowledge of Bethe ansatz equations and eigenvalues of transfer matrix explicitly, but gives a possibility of finding basic physical quantities of the model quite easily.

Plan of the paper is the following. We start section 2 with the description of the Boltzmann weights of the models in terms of multicoloured TL generators and their algebraic relations. We analyse possible physical regimes, where Boltzmann weights are real and positive, and discuss possible vertex and IRF interpretation of the models. Some interesting factorization property of admissibility graph for IRF models is pointed out. In section 3, we derive functional inversion relations for the transfer matrices of two-coloured vertex model. We solve them analytically as a product of analytical non-zero function, and a function with poles or zeros. Excited states are discussed and found to lead to a mass gap. In section 4, the generalization of inversion relations for multicoloured models is discussed. We conclude by section 5 with a discussion of the obtained results.

2. Coloured TL lattice models

In this section, we recall the results of [1] and demonstrate them for few examples of two-, three- and four-coloured models. For details of Fuss–Catalan algebras and their representations we refer the reader to [2] and [1]. We also discuss vertex and IRF representations of baxterized Fuss–Catalan models and find their possible physical regimes.

2.1. Solutions of Yang–Baxter equation with Fuss–Catalan symmetry

New class of TL-like algebras found by Bisch and Jones [2] can be considered as a kind of quotient of tensor product of few TL algebras. Generators of these algebras may be formulated in terms of TL generators defined on a lattice. The relations on generators depend on the parity of the site i where the operator acts. The main commutation relations [2] are the following:

$$U_i^{(m)}U_i^{(p)} = U_i^{(p)}U_i^{(m)} = \rho_i(\min(m,p))U_i^{(\max(m,p))} \tag{1}$$

where

$$\rho_i(p) = \begin{cases} \alpha_1\alpha_2\cdots\alpha_p & i \text{ even} \\ \alpha_k\alpha_{k-1}\cdots\alpha_{k+1-p} & i \text{ odd} \end{cases}. \tag{2}$$

Here, k is the number of colours, and in the standard strings pictorial interpretation of the operators $U^{(m)}$ weight α_m is attached to each loop of colour m. Locality of operators U is expressed as

$$U_i^{(m)}U_j^{(p)} = U_j^{(p)}U_i^{(m)} \tag{3}$$

when $|i-j| > 1$ or when $|i-j| = 1$ and $m+p \leqslant k$. Non-commuting operators satisfy cubic relations

$$U_i^{(m)}U_{i\pm1}^{(p)}U_i^{(q)} = \rho_i(k-p)\begin{cases} U_i^{(m)}U_{i\pm1}^{(k-q)} & m \geqslant q \\ U_{i\pm1}^{(k-m)}U_i^{(q)} & m \leqslant q. \end{cases} \tag{4}$$

These are basic relations of the Fuss–Catalan algebra $FC_{k(N+1)}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ which describes coils of $k(N+1)$ strings of k colours. Boundary conditions are fixed by requirement that both on the top and on the bottom strings end up by the same pattern $a_1 a_2 \cdots a_k a_k a_{k-1} \cdots a_2 a_1 a_1 \cdots$. Pictorial interpretation of the commutation relations written above one can find in [1]. Let us note that not all of the relations written above are independent (for details see [1, 2]). There are some necessary consequences of these relations, in particular, if one wishes to check the validity of Yang–Baxter equations for the baxterization solutions, but we will not stop on these technical details here.

In [1], baxterization of this algebra was successfully done. Namely, all possible solutions of the Yang–Baxter equation

$$W_i(x)W_{i+1}(xy)W_i(y) = W_{i+1}(y)W_i(xy)W_{i+1}(x) \tag{5}$$

were found in the form

$$W_i(x) = 1_i + \sum_{m=1}^{k} a_m(x) U_i^{(m)} \tag{6}$$

satisfying

$$W_i(1) = 1_i \tag{7}$$

in the class of rational polynomial functions $a_m(x)$. It was done by the substitution of (6) into (5), which, using the Fuss–Catalan algebra, leads to a complicated system of functional equations on the coefficients $a_m(x)$. General solution of this system of functional equations was found to be

$$a_m(x) = \frac{1}{\rho_m} x^{r_1+r_2+\cdots+r_{m-1}}(x^{r_m} - 1), \qquad m = 1, 2, \ldots, k-1. \tag{8}$$

$$a_k(x) = \frac{\rho_1}{\rho_{k-1}} x^{r_1+r_2+\cdots+r_{k-2}+1} \frac{x^{r_{k-1}} - 1}{\mu - x}, \tag{9}$$

where $\rho_m \equiv \rho_i(m)$ are defined in (2), $r_m = \{\pm 1\}$, and r_m and other parameters entering the last equations are restricted and defined by the following set of relations: $r_1 = r_{k-1} = 1$ and either $r_{q-1} = r_q = r_{k-q} = r_{k-q+1}$, or $r_{q-1} = -r_q = -r_{k-q} = r_{k-q+1}$, and

$$\mu^{(r_{q-1}+r_q)/2} = \alpha_q^2, \qquad \mu = \alpha_1^2 - 1, \qquad \alpha_q = \alpha_{k+1-q} \tag{10}$$

for any $q = 2, \ldots, k-1$. In [1], the special solution with $r_1 = r_2 = \cdots = r_{k-1} = 1$ was called fundamental. With this set of r_q one has $\mu = \alpha_1^2 - 1 = \alpha_2^2 = \cdots = \alpha_{k-1}^2$. These equations were solved in the assumption that all α's are positive. We will relax this condition and permit two possible branches of the square root when solving (10) with respect to α_q. For real α_q it means the option of some negative α's will not be ignored. We introduce the set of signs of α_q, $s_q = \{\pm 1\}$, $q = 2, \ldots, k-1$, such that $s_q = s_{k+1-q}$. Then, we get the solutions in the following general form:

$$a_q(x) = \left(\prod_{j=2}^{q} s_j\right)^{q-1} \left(\frac{x}{\sqrt{\alpha^2 - 1}}\right)^{\sum_{j=1}^{q-1} r_j} \frac{(x^{r_q} - 1)}{\alpha}, \qquad q = 2, \ldots, k-1, \tag{11}$$

$$a_k(x) = \left(\prod_{j=2}^{k-1} s_j\right)^{k-2} \left(\frac{x}{\sqrt{\alpha^2 - 1}}\right)^{\sum_{j=1}^{k-2} r_j} \frac{x(x-1)}{\alpha^2 - 1 - x}. \tag{12}$$

Note that an important artefact of the baxterization procedure in the case of the Fuss–Catalan algebras is a necessary additional requirement that all the parameters α_q of different copies of TL subalgebras should be equal (up to some signs). Here and below we denote α_1 as α—the only free parameter remaining after baxterization, which we in principle permit to be negative. This possibility of negative α(s) will extend variety of possible physical regimes of our statistical models.

In [1], the solutions with $s_q = r_q = 1$ were called fundamental. All others differ from it either by less powers of x appearing in a_q or alternating signs of a_q. Both types of changes compared to the fundamental solution should respect the symmetries of parameters r and (10). The fundamental solution can be written as

$$W_i(x) = 1_i + \sum_{q=1}^{k-1} \frac{x^{q-1}}{(\sqrt{\alpha^2-1})^{q-1}} \frac{x-1}{\alpha} U_i^{(q)} + \frac{x^{k-1}}{(\sqrt{\alpha^2-1})^{k-2}} \frac{x-1}{\alpha^2-1-x} U_i^{(k)}. \tag{13}$$

All the solutions for Boltzmann weights (6) we got have two important properties: unitarity

$$W_i(x) W_i(1/x) = 1_i \tag{14}$$

and crossing symmetry

$$\overline{W}_i(x_*^2/x) = \left(\frac{x_*}{x}\right)^{\sum_{j=1}^{k-2} r_j} \frac{x_*^2 - x}{x(x-1)} W_i(x) \tag{15}$$

which becomes

$$\overline{W}_i(x_*^2/x) = \left(\frac{x_*}{x}\right)^{k-2} \frac{x_*^2 - x}{x(x-1)} W_i(x) \tag{16}$$

for the fundamental models. Here $x_* = \sqrt{\alpha^2 - 1}$, the 'bar' operation means $\overline{U}_i^{(q)} = U_i^{(k-q)}, q = 1, \ldots, k \left(U_i^{(0)} \equiv 1_i\right)$. In pictorial representation 'bar' means rotation of TL generators by 90°, as in the crossing relation of Boltzmann weights of a lattice model. These two symmetries (14), (15) are very important and play crucial role in derivation of inversion relation. Here and below we mainly stop on few representative examples of the many solutions described by (11)–(13). We will analyse in details $k = 2, 3, 4$, considering possible regimes of the models.

2.2. Examples and regimes

Here we explicitly write all the solutions found for $k = 2, 3, 4$. Two-colour case has only fundamental solution

$$W_i^{(2)}(x) = 1_i + \frac{x-1}{\alpha} U_i^{(1)} + \frac{x(x-1)}{\alpha^2 - 1 - x} U_i^{(2)}. \tag{17}$$

In the case of three colours there are two possibilities

$$W_i^{(3\pm)}(x) = 1_i + \frac{x-1}{\alpha} U_i^{(1)} \pm \frac{x(x-1)}{\alpha\sqrt{\alpha^2-1}} U_i^{(2)} \pm \frac{x^2(x-1)}{\sqrt{\alpha^2-1}\left(\alpha^2-1-x\right)} U_i^{(3)} \tag{18}$$

where the case of upper signs (3+) corresponds to what we called the fundamental solution. There are more possibilities in the case of four colours. It is the minimal number of colours when what we called an excited solution (some $r_q = -1$) exists. Two solutions are

$$W_i^{(4\pm)}(x) = 1_i + \frac{x-1}{\alpha} U_i^{(1)} \pm \frac{x(x-1)}{\alpha\sqrt{\alpha^2-1}} U_i^{(2)} + \frac{x^2(x-1)}{\alpha(\alpha^2-1)} U_i^{(3)} + \frac{x^3(x-1)}{(\alpha^2-1)(\alpha^2-1-x)} U_i^{(4)}. \tag{19}$$

The fundamental solution is (4+) and (4−) is an excited one. There are more four-colour excited solutions:

$$W_i^{(4'\pm)}(x) = 1_i + \frac{x-1}{\alpha}U_i^{(1)} \pm \frac{x-1}{\alpha}U_i^{(2)} + \frac{x-1}{\alpha}U_i^{(3)} + \frac{x(x-1)}{(\alpha^2-1-x)}U_i^{(4)}. \quad (20)$$

If we now impose positivity of all Boltzmann weights, we can find different regimes of the models. It is convenient to give names to the following regions on the plain (α, x):

$$\begin{array}{lll} \text{I}: & 1 < x < \alpha^2 - 1, & \sqrt{2} < \alpha, \\ \text{II}: & \alpha^2 - 1 < x < 1, & -\sqrt{2} < \alpha < -1, \\ \text{III}: & 0 < x < 1, & -1 < \alpha < 0, \\ \text{IV}: & -\infty < x < -1, & -\infty < \alpha < -1, \\ \text{V}: & -\infty < x < \alpha^2 - 1, & -1 < \alpha < 0. \end{array} \quad (21)$$

We will use the following parameterization of spectral and crossing parameters in the regions I–V:

$$\text{I, II}: \quad x = \mathrm{e}^u, \quad \alpha^2 - 1 = \mathrm{e}^\lambda, \quad \text{I}: \quad 0 < u < \lambda, \quad \text{II}: \quad \lambda < u < 0. \quad (22)$$

$$\text{III}: \quad x = \mathrm{e}^u, \quad \alpha^2 - 1 = -\mathrm{e}^\lambda, \quad \lambda < 0, \quad u < 0 \quad (23)$$

$$\text{IV}: \quad x = -\mathrm{e}^u, \quad \alpha^2 - 1 = \mathrm{e}^\lambda, \quad 0 < u \quad (24)$$

$$\text{V}: \quad x = -\mathrm{e}^u, \quad \alpha^2 - 1 = -\mathrm{e}^\lambda, \quad \lambda < 0, \quad \lambda < u. \quad (25)$$

Models ‘2’ and ‘4′+’ are physical (have positive Boltzmann weights) in all the regimes I–V. ‘3+’ is physical in the regimes I and II, and ‘3−’ in the regimes IV, V. ‘4+’ has all Boltzmann weights positive only in the regime I, whereas models ‘4−’ and ‘4′−’ are never physical in the sense of all Boltzmann weights positivity.

We see a rich spectrum of possibilities for different regions of parameters defining a positive Boltzmann weights. Only detailed investigation of the models can answer the question what is the phase structure of the models and its relation to the regimes.

2.3. Vertex and IRF representations

In principle, it would be plausible to proceed with exact solution of these new integrable lattice models directly in terms of Fuss–Catalan generators. Such a possibility is shortly discussed in section 5. Here and below we formulate integrable lattice models in terms of more concrete representations of the Fuss–Catalan algebra. In this context, the useful fact [2] is that the Fuss–Catalan algebra (1), (3) and (4) is a subalgebra of tensor product of few copies of TL algebras with different parameters $TL(\alpha_1) \otimes \cdots \otimes TL(\alpha_k)$, and generators $U_i^{(q)}$ may be expressed in terms of $u_i^{(q)} \in TL(\alpha_q)$ as

$$U_i^{(q)} = \begin{cases} 1 \otimes \cdots \otimes 1 \otimes u_i^{(k-q+1)} \otimes \cdots \otimes u_i^{(k)} & i \text{ odd} \\ u_i^{(1)} \otimes \cdots \otimes u_i^{(q)} \otimes 1 \otimes \cdots \otimes 1 & i \text{ even.} \end{cases} \quad (26)$$

Recall that one of the requirements of the baxterization procedure was the same value of $|\alpha_q|$ for different q.

The most studied lattice models are vertex models and interaction round the face (IRF) models. In the first case, the TL operator $U_i^{(q)}$ acts on linear spaces attached to four links of square lattice connected to each site. Using a basis $\{e_i\}$ of the linear space, the vertex Boltzmann weights W is defined as

$$W(x)e \otimes e = \sum_{i,j} W_{ij}^{kl}(x) e_i \otimes e_j. \tag{27}$$

The fundamental representation for the usual TL generator u is known to be defined on a tensor product of two-dimensional linear spaces in a matrix form:

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 \\ 0 & 1 & 1/z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

where z is related to the TL algebra parameter $\alpha = z + 1/z$. In view of (26), it is natural to define representation for the multicoloured generator $U_i^{(q)}$ as the tensor product of matrices (28) (with, in principle, different parameters z) and identity matrices, as in (26). The linear spaces living on the links of the lattice are of dimension 2^k, where k is the number of colours.

Another representation of algebraically formulated lattice model Boltzmann weights $W_i(x)$ is an IRF representation. In this case, some (integer) numbers are living at each site of the square lattice with a constraint that on two admissible (connected by a link) sites may appear only numbers permitted by some fixed admissibility graph. Then, the representation is given (see, e.g., [11]) on an elementary face of the lattice with site values (i, j, k, l) in the anticlockwise direction:

$$1_{ij}^{lk} = \delta_{ik}, \qquad u_{ij}^{lk} = \frac{\sqrt{S_i S_k}}{S_j} \delta_{jl} \tag{29}$$

where S_i are components of the eigenvector of the admissibility matrix with the largest eigenvalue equal to the TL parameter α. $\alpha = 2\cos\frac{\pi}{m}$, $S_i = \sin\frac{\pi i}{m}$, with m-Coxeter number for models with ADE-like admissibility graphs, and $\alpha = 2$ for the extended $\widehat{A}\widehat{D}\widehat{E}$ admissibility graphs. Further construction of representation of $U_i^{(q)}$ generators goes by tensoring (29) according to (26). This tensor product of elementary IRF representations of TL operators will be defined on a tensor product admissibility graph. As we said above, not all admissibility graphs may participate in this tensor product, since their matrix eigenvalues are not arbitrary, but restricted by condition (10). Possible partners among ADE models which can be tensored were classified in [1]. Actually, there is even more freedom—one can make a tensor product of vertex and IRF models, consistently fitting their parameters α. But in what follows we will concentrate on two simplest examples—two-colour vertex model and two-colour A_n IRF models.

The simplest possibility for vertex realization is the model '2'. It has 32 non-zero Boltzmann weights and one can find their list in [1]. Looking at this list, one can see that the model is not of ice type—there is no a natural 'charge' conservation in each vertex. Linear space living on each of four edges of a vertex is four dimensional, such that R-matrix is 16×16. An interesting observation is that it is possible to attach 'charge' to these four spaces in a way preserving 'charge' conservation, but at a cost of charge degeneracy of the spaces. For instance, the attachment—space 0 → charge 0, space 1 → charge 1, space → charge 1, space 3 → charge 3—leads to 'charge' conservation in each non-zero vertex.

We formulate the model in the language of spin chain models and consider the Boltzmann weights W as an L-operator. (Recall that R-matrix is related to L-operator by permutation P of outgoing spaces: $R = PL$ or $R_{ij}^{kl} = W_{ji}^{kl}$). We define the transfer matrix as

$$T_{o(e)}(x) = (W_{o(e)}(x))_{c_N b_1}^{a_1 c_1} (W_{e(o)}(x))_{c_1 b_2}^{a_2 c_2} (W_{o(e)}(x))_{c_2 b_3}^{a_3 c3} \cdots (W_{e(o)}(x))_{c_{N-1} b_N}^{a_{N-1} c_N}. \tag{30}$$

There is a summation over repeated indices here, and indices o/e mean oddness(evenness) of the south site along the line of the auxiliary space $c\ldots$. As we see, T is an alternating product of Boltzmann weights on even and odd sites. (Recall that in the multicoloured TL models they are different). In the language of spin chains Boltzmann weights may be considered as L-operators. Unitarity condition looks like

$$(W_{o(e)}(x))^{ab}_{cd}(W_{o(e)}(x^{-1}))^{cd}_{ef} = \delta^a_e\delta^b_f. \tag{31}$$

There is an important property of BW, which is a mixture of crossing (15), (16) and unitarity relation:

$$W_{o(e)}(x) = \frac{x(x-1)}{x_*^2 - x} M^{(1)}[W_{e(o)}(x_*^2/x)]^{t_2} M^{(1)}. \tag{32}$$

Here t_2 means transpose in the second space of Boltzmann weights, corresponding to the auxiliary space of the transfer matrix. The matrix $M^{(1)}$ acts in the first space and has the form

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1/z & 0 \\ 0 & 1/z & 0 & 0 \\ 1/z^2 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

Another important symmetry is

$$(W_{o(e)}(x))^{ab}_{cd} = P W_{e(o)}(x) P = (W_{e(o)}(x))^{ba}_{dc}. \tag{34}$$

Relations (31), (32) are almost enough for the derivation of the inversion relation (see below).

As the simplest example of two-colour IRF models one can consider the case of tensor product admissibility graph $A_n \times A_n$. The general form of the RSOS Boltzmann weights is defined by (17), (26), (29):

$$W_{e(o)}(ii', jj', kk', ll'|x) = \delta_{ik}\delta_{i'k'} + \frac{x-1}{\alpha}\begin{cases} \delta_{i'k'}\delta_{jl}\frac{\sqrt{S_i S_k}}{S_j} & \text{even site} \\ \delta_{ik}\delta_{j'l'}\frac{\sqrt{S_{i'}S_{k'}}}{S_{j'}} & \text{odd site} \end{cases} \tag{35}$$
$$+ \frac{x(x-1)}{\alpha^2 - 1 - x}\delta_{jl}\delta_{j'l'}\frac{\sqrt{S_i S_k S_{i'} S_{k'}}}{S_j S_{j'}}.$$

Pairs of variables (i, i'), (j, j'), (k, k'), (l, l'), sitting at the south, east, north and west corners of a face, are any admissible set of integers in the range from 1 to n. Each of two quartets (i, j, k, l) and (i', j', k', l') are those permitted by A_n admissibility graph diagram independently, and $S_k = \sin\frac{\pi k}{n+1}$, $\alpha = 2\cos\frac{\pi}{n+1}$. One can explicitly check that these Boltzmann weights satisfy the following unitarity and crossing relations:

$$\sum_{k,k'} W_{e(o)}(ii', jj', kk', ll'|x) W_{o(e)}(kk', jj', mm', ll'|1/x) = \delta_{im}\delta_{i'm'} \tag{36}$$

$$W_{e(o)}(ii', jj', kk', ll'|x_*^2/x) = \frac{\alpha^2 - 1 - x}{x(x-1)}\sqrt{\frac{S_j S_l S_{j'} S_{l'}}{S_i S_k S_{i'} S_{k'}}} W_{o(e)}(jj', kk', ll', ii'|x). \tag{37}$$

An interesting and important feature of the Boltzmann weights (35) is their admissibility graph decomposition property. One can check that admissibility rules coming from (35) lead to two independently existing subsets of site variables, each with its own admissibility graph. The form of these two graphs depends on oddness of n (see figures 1(a) and (b)).

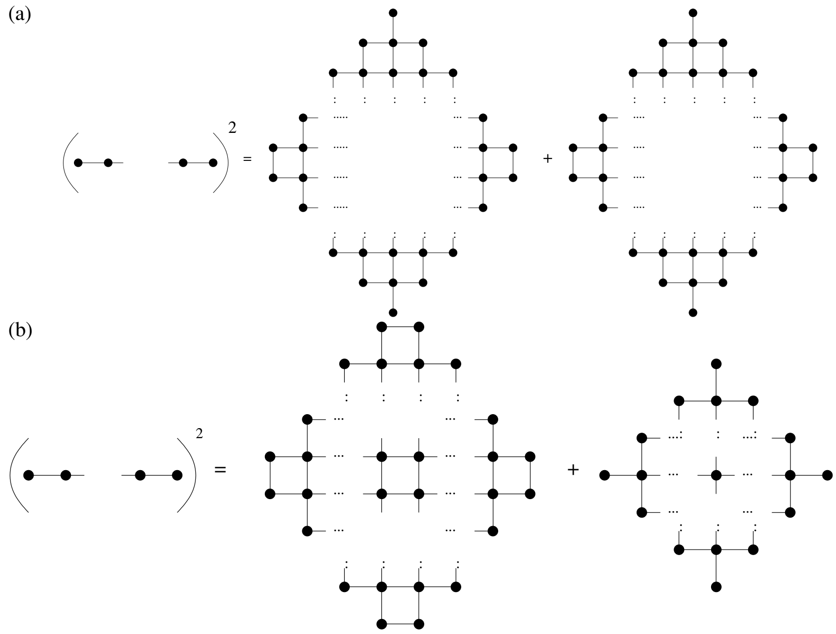

Figure 1. $(A_n)^2$ admissibility graph decomposition: (*a*) n is even (upper) and (*b*) n is odd (lower).

Again, the basic object of investigation will be the transfer matrix

$$T_{o(e)}(x)^{\{\beta\beta'\}}_{\{\alpha\alpha'\}} = \prod_{i=1}^{N/2} W_{o(e)}(\alpha_i\alpha'_i, \alpha_{i+1}\alpha'_{i+1}, \beta_{i+1}\beta'_{i+1}, \beta_i\beta'_i|x) \\ \times W_{e(o)}(\alpha_{i+1}\alpha'_{i+1}, \alpha_{i+2}\alpha'_{i+2}, \beta_{i+2}\beta'_{i+2}, \beta_{i+1}\beta'_{i+1}|x) \tag{38}$$

where the trace identification $\alpha_0 = \alpha_N$, $\alpha'_0 = \alpha'_N$, $\beta_0 = \beta_N$, $\beta'_0 = \beta'_N$ is imposed. Recall here an important and well-known difference in the definition of TM for vertex and IRF cases. In the vertex case, the auxiliary space of TM contains a sum over thermodynamically many (N) variables c_i in (30), whereas the IRF TM does not contain any auxiliary space.

Let us look in more details at the two-colour $(A_3)^2$ graph IRF model. This model, in a sense, may be called A_3'two-colour' Ising model, since, as it is well known (see, e.g., [12]), A_3 graph IRF model is equivalent to the Ising model. Recall that A_3 admissibility rule requires that the height value is fixed and equal to 2 (the value of the middle site of A_3 graph) on the even sublattice of two-dimensional square lattice, whereas the heights on the odd sublattice take the values 1 and 3, forming an Ising model. The situation is different in the two-colour case and the model seems to be more complicated then Ising model. As one can see by explicit analysis of non-zero Boltzmann weights for the case $n = 3$, the RSOS model splits into two subsets of variables living on the sites of a face. If one denotes ($\{\overline{\alpha}\}$) the pairs of two numbers at the sites of a face as

$$(ij) \equiv \overline{\alpha} = 3(i-1) + j, \tag{39}$$

the decomposition of the admissibility graph into two looks like it is shown in figure 2. It means that there are either even or odd heights in the new notation, living on each site of the model. It means the model factorizes into two separate submodels.

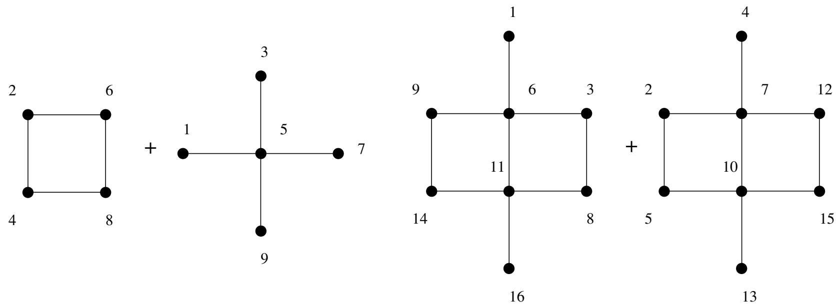

Figure 2. Admissibility graph decomposition for $(A_3)^2$ and $(A_4)^2$: (*a*) $n = 3$ and (*b*) $n = 4$.

In the same way, the admissibility graph of the two-colour $(A_4)^2$ graph IRF model, after the recounting of the site variables

$$(ij) \equiv \overline{\alpha} = 4(i-1) + j, \tag{40}$$

decomposes into two disconnected graphs (see figures 2(*a*) and (*b*)), dividing the model into two submodels—one with the site variables $\overline{\alpha} = \{1, 3, 6, 8, 9, 11, 14, 16\}$ other with $\overline{\alpha} = \{2, 4, 5, 7, 10, 12, 13, 15\}$. Again there are two separate submodels.

Note that relations (37) are valid for each of two submodels of $(A_n)^2$ admissibility graph decomposition.

3. Inversion functional relations and their solutions

The most full and contemporary investigation of the models described in the previous section should probably follow the standard methods of algebraic Bethe ansatz. Unfortunately, the obstructions we met on this way did not allow us to proceed in this direction, and algebraic Bethe ansatz for these models remains an open problem. But some information about ground-state energy and free energy, spectrum of low-lying excitations, and even about some correlation functions, can be extracted using a method known for a long time [7], which does not require a knowledge of explicit form BA equations and transfer matrix eigenvalues. It is a method of inversion functional relations on the transfer matrix eigenvalues, which was successfully applied to few integrable lattice models [10]. It would be interesting to develop a universal algebraic method of derivation of inversion relations in terms of multicoloured TL algebra generators, as it was partly done in [13] for usual TL algebra models, but in what follows we are going to consider the inversion relation in vertex representations of multicoloured TL algebra, which we discussed in the previous section.

3.1. Ground state of vertex model

Consider the product

$$(M)^{\otimes N} T_o(x_*^2 x)(M)^{\otimes N} T_e(x). \tag{41}$$

Since transfer matrices commute, their eigenvalues will satisfy the same functional relations as transfer matrices themselves. The above expression explicitly looks as

$$Tr_A Tr_B \big[M^{C_1} W^{t_A}_{oAC_1}(x_*^2 x) M^{C_1} M^{C_2} W^{t_A}_{eAC_2}(x_*^2 x) M^{C_2} \cdots * W_{eBC_1}(x) W_{oBC_2}(x) \cdots \big].$$

Insertion of the equal to 1 factor $(\delta_{a_ib_i} + 1 - \delta_{a_ib_i})$ in any place of this product leads to a split of the above expression into two terms—the first corresponding to $\delta_{a_ib_i}$ and the second to $(1 - \delta_{a_ib_i})$. Due to relations (32), (31) the first term turns out to be proportional to the identity operator in all the C_i spaces. Oppositely, the second term is not giving an identity operator in any of C_i spaces. According to (32), the proportionality coefficient for the first term will be $\frac{1}{z^{2N}}\left(\frac{x_*^2-x^{-1}}{x^{-1}(x^{-1}-1)}\right)^N$. One can show that the expansion of the second term in powers of x starts from $x^{N/2}$, therefore in the region where $x > 1$, e.g. in the regime I, the second term is exponentially small compared to the first term in the thermodynamic limit. This leads to the following inversion relation:

$$(M)^{\otimes N} T_o(x_*^2 x)(M)^{\otimes N} T_e(x) = \frac{1}{z^{2N}} \left(\frac{x_*^2 - x^{-1}}{x^{-1}(x^{-1} - 1)} \right)^N I_{C_1} \otimes I_{C_2} \otimes \cdots \otimes I_{C_N} + O(\mathrm{e}^{-N}). \tag{42}$$

One can also see that the specificity of the matrix M is such that the relation between highest eigenvalue of $(M)^{\otimes N} T_o(x_*^2 x)(M)^{\otimes N}$ and of $T_o(x_*^2 x)$ is by factor $\frac{1}{z^{2N}}$. It leads to the following functional relation for eigenvalues of transfer matrix in the thermodynamic limit $N \to \infty$, when the terms $O(\mathrm{e}^{-N})$ can be neglected,

$$\Lambda(x_*^2 x)\Lambda(x) = \left(\frac{x_*^2 - x^{-1}}{x^{-1}(x^{-1} - 1)} \right)^N. \tag{43}$$

Another relation is the unitarity relation. If one considers the product

$$T_o(x)T_e(1/x) = Tr_A Tr_B [W_{oAC_1}(x) W_{eAC_2}(x) \cdots W_{eBC_1}(1/x) W_{oBC_2}(1/x) \cdots] \tag{44}$$

then by insertion of the space permutation operator P $(P^2 = 1)$ between the Boltzmann weights of, say, the second TM, and using (34), one gets

$$\sum_{\{a\},\{b\},\{c\}} W_o(x)_{a_N c_1}^{d_1 a_1} W_e(x)_{a_1 c_2}^{d_2 a_2} \cdots W_o(1/x)_{e_1 b_N}^{b_1 c_1} W_e(1/x)_{e_2 b_1}^{b_2 c_2} \cdots. \tag{45}$$

If one inserts one delta symbol $\delta_{e_{i+1}a_i}$ for some i, then due to relation (31) all the product will be reduced to the identity operator $\mathrm{Id} = \delta_{e_1 d_1} \cdots \delta_{e_N d_N}$. One can also show that other terms successfully cancel. This leads to the second functional relation on the transfer matrix eigenvalues:

$$\Lambda(x)\Lambda(1/x) = 1. \tag{46}$$

The functional equations (43), (46) can be solved by standard methods. These equations in the parameterization (22) for regime I can be written in terms of $\overline{\Lambda}(u) = \Lambda^{1/N}(x)$ as

$$\begin{aligned} \overline{\Lambda}(u)\overline{\Lambda}(\lambda + u) &= \frac{\mathrm{e}^{\lambda+u} - 1}{\mathrm{e}^{-u} - 1} \\ \overline{\Lambda}(u)\overline{\Lambda}(-u) &= 1. \end{aligned} \tag{47}$$

In order to uniquely solve this system of functional equations, one needs additional information about analyticity of the function $\overline{\Lambda}(u)$. One can look for solutions in the form $\overline{\Lambda}(u) = F(u)f(u)$, where $F(u)$ is a general analytic non-zero (ANZ) in the quadrant $0 < \mathrm{Re}(u) < \lambda, 0 < \mathrm{Im}(u) < 2\pi$ solution of the system

$$\begin{aligned} F(u)F(\lambda + u) &= \frac{\mathrm{e}^{\lambda+u} - 1}{\mathrm{e}^{-u} - 1} \\ F(u)F(-u) &= 1 \end{aligned} \tag{48}$$

and $f(u)$ is a solution of

$$\begin{aligned} f(u)f(u+\lambda) &= 1 \\ f(u)f(-u) &= 1 \end{aligned} \tag{49}$$

with a given set of poles u_p and zeros u_z in the same quadrant. We have cut the strip $0 < \mathrm{Im}(u) < 2\pi$ in the imaginary direction because of the obvious $2\pi\mathrm{i}$ periodicity of solutions of the system (47). The cut in the real axis direction $0 < \mathrm{Re}(u) < \lambda$ is a consequence of the relation $f(u+2\lambda) = f(u)$, which one can easily derive from (49).

The general ANZ solution of (48) has the form

$$F(u) = (\mathrm{e}^u - 1)\prod_{j=0}^{\infty}\frac{(\mathrm{e}^{(2j+1)\lambda+u}-1)(\mathrm{e}^{(2j+2)\lambda-u}-1)}{(\mathrm{e}^{2j\lambda+u}-1)(\mathrm{e}^{(2j+1)\lambda-u}-1)}. \tag{50}$$

For the double periodic function $f(u)$ we have the system of functional equations (49). Provided we know the set of its poles and zeros in the periodicity quadrant, the solution is uniquely fixed by Liouville's theorem:

$$f(u) = \pm\prod_{u_z}\sqrt{\kappa_1}\,\mathrm{snh}\left(\frac{2K_1}{\pi}(u-u_z)\right)\prod_{u_p}\sqrt{\kappa_1}\,\mathrm{snh}\left(\frac{2K_1}{\pi}(u-u_p-\lambda)\right) \tag{51}$$

where snh is a standard elliptic function with modulus κ_1 defined by the requirement that corresponding quarter periods K_1, K_1' are related by $\frac{K_1'}{K_1} = \frac{\lambda}{\pi}$. Equations (50) and (51) give general form of transfer matrix highest eigenvalue as a function of spectral parameter u, provided one knows positions of its poles and zeros in the periodicity region. Obviously, transfer matrix eigenvalues do not have poles in the physical region. Some numerics we did for the lattices of small size in regime I support the conjecture that the ground-state eigenvalue does not also have zeros in the periodicity region, and $f(u) = 1$ for the ground state. Using (50) one can calculate the ground-state energy of the model:

$$E_0 = -\frac{\mathrm{d}}{\mathrm{d}u}(\ln\overline{\Lambda}(u))_{u=0} = -\sum_{j=0}^{\infty}\frac{\sinh\frac{\lambda}{2}}{\sinh(j+1)\lambda\,\sinh\left(j+\frac{1}{2}\right)\lambda}. \tag{52}$$

3.2. *Spectrum of low-lying excitations*

In the same way one can consider the low-lying excitations, i.e. those states for which the transfer matrix eigenvalue $\Lambda(u)$ is finitely different from the ground state $\Lambda_{\mathrm{gs}}(u)$ in the thermodynamic limit. For instance, define finite function

$$l(u) = \lim_{N\to\infty}\frac{\Lambda(u)}{\Lambda_{\mathrm{gs}}(u)} \tag{53}$$

for vertex model. Again $l(u)$ can be represented as a product of ANZ function $F(u)$ and another one $f(u)$ satisfying the same functional equations (48), (49). But this time $f(u) \neq 1$, and some zero(s) u_z enter into the periodicity domain. The excitation spectrum can be calculated from the relation between energy and momentum of excitations:

$$\begin{aligned} E - E_0 &= -\sum_{\{u_z\}}\frac{\mathrm{d}}{\mathrm{d}u}(\ln\psi(u))_{u=0} = \sum_{\{u_z\}}\varepsilon(p(u_z)) \\ P - P_0 &= -i\sum_{\{u_z\}}\ln\psi(0) = \sum_{\{u_z\}}p(u_z) \end{aligned}$$

where $\ln\psi(u)$ is the contribution of each snh factor in the product (51). Using (51) we get the dispersion relation

$$\varepsilon(p) = K'\sqrt{(1-\kappa)^2 + 4\kappa\sin^2 p} \tag{54}$$

with non-zero mass gap

$$\Delta = K'(1-\kappa). \tag{55}$$

So, at least in the regime I, where one can neglect the additional terms in the inversion relation, all the excitations are with a non-zero energy gap. It means there is no conformal critical point in this regime.

Importance of the regime in all what was done above is crucial in two aspects. First, in order to neglect the non-identity terms in the inversion relation one needs $|x| > 1$, which is correct not only for regime I, but also for regime IV and $x < -1$ subregion of the regime V (we will call it V′). Second, the numerics, which is necessary to be sure in ANZ condition, essentially differ for different regimes. The analogue of (47) for vertex model in a parameterization (24), (25) of the regimes IV, V′ looks like

$$\begin{aligned}\overline{\Lambda}(u)\overline{\Lambda}(\lambda+u) &= \frac{e^{\lambda+u}-p}{e^{-u}+1}\\ \overline{\Lambda}(u)\overline{\Lambda}(-u) &= 1\end{aligned} \tag{56}$$

where in the regime IV $p=-1$ and in the regime V′ $p=1$. Equation (49) remains as it is. It means the solution for F in this case is given by

$$F(u) = (e^u+1)\prod_{j=0}^{\infty}\frac{(e^{(2j+1)\lambda+u}-p)(e^{(2j+2)\lambda-u}+1)}{(e^{2j\lambda+u}+1)(e^{(2j+1)\lambda-u}-p)} \tag{57}$$

$f(u)$ is given by the same solution (51). With the assumption of ANZ condition for $\Lambda(u)$, it leads to the following ground-state energies of the two-colour vertex model in the regimes IV and V′:

$$E_0^{\mathrm{IV}} = -\sum_{j=0}^{\infty}\frac{\cosh\frac{\lambda}{2}}{\cosh(j+1)\lambda\cosh\left(j+\frac{1}{2}\right)\lambda}, \qquad E_0^{\mathrm{V}'} = -\sum_{j=0}^{\infty}\frac{\cosh\frac{\lambda}{2}}{\cosh(j+1)\lambda\sinh\left(j+\frac{1}{2}\right)\lambda}. \tag{58}$$

The analysis of low-lying excitations remains the same, giving the same finite energy gap (55).

4. Other models

A natural conjecture is that all what was done in the previous section for two-coloured models can be generalized for what was called above fundamental models with any number of colours in the regime analogous to regime I. Let us point out that both three- and four-coloured models have only regimes I and II as regimes with all Boltzmann weights positive. As we said, $x > 1$ only in the regime I, which was crucial for the derivation of inversion relation. Instead of (32) we have

$$L_i(x) = \left(\frac{x_*}{x}\right)^{k-2}\frac{x(x-1)}{x_*^2-x}\widetilde{M}^{(1)}[L_{i+1}(x_*^2/x)]^{t_2}\widetilde{M}^{(1)} \tag{59}$$

with some matrix $\widetilde{M}$ acting in the first space. Together with (31) it gives the following inversion relations:

$$\begin{aligned}T(x)T(x_*^2x) &= (x_*x)^{k-2}\left[\frac{x_*^2-x^{-1}}{x^{-1}(x^{-1}-1)}\right]^N \mathrm{Id} + O(e^{-N})\\ T(x)T(1/x) &= \mathrm{Id}.\end{aligned}$$

As we see, (59) differs from (42) in a minimal way—by a prefactor. The same procedure as previously gives

$$\Lambda(u) = \overline{\Lambda}(u) f(u)$$
$$\overline{\Lambda}(u) = \mathrm{e}^{(k-2)u}(\mathrm{e}^{u} - 1)\prod_{j=0}^{\infty}\frac{(\mathrm{e}^{(2j+1)\lambda+u} - 1)(\mathrm{e}^{(2j+2)\lambda-u} - 1)}{(\mathrm{e}^{2j\lambda+u} - 1)(\mathrm{e}^{(2j+1)\lambda-u} - 1)}$$

with the same expression (51) for $f(u)$. It leads to integer shift of the ground-state energy in $k-2$, compared to (52), provided still $f = 1$ (ANZ hypothesis). This of course should be checked numerically.

Looking at the other models which have some physical regimes, one can believe that the general form of inversion relations both for fundamental models with some negative signs and for excited models with or without negative signs is just

$$T(x)T(x_*^2 x) = (x_* x)^{\sum r_j - 2}\left[\frac{x_*^2 - x^{-1}}{x^{-1}(x^{-1} - 1)}\right]^N \mathrm{Id} + O(\mathrm{e}^{-N})$$
$$T(x)T(1/x) = \mathrm{Id}.$$

Form of solution of these equations depends on regime. We see that different models have the same set of functional inversion relations and sometimes even the same set of regimes.

One can believe that if there is a difference in ground-state or low-lying excited states between these models, then it is expressed in the amount and location of poles and zeros of function $f(u)$. An information about these poles and zeros may come from numerics.

5. Discussion

In this paper, we made an attempt to analyse vertex and IRF multicoloured lattice models. We saw an interesting structure of admissibility graph for IRF models. Being unable to solve the models by algebraic Bethe ansatz, we tried to analyse them using the inversion relations. The main result of this attempt—all the vertex models have a mass gap in the considered regime. We can try to construct new link invariants, following a standard procedure of extracting a braid group representation from the Boltzmann weights either in vertex or in IRF form (see [14] and references therein). The objection we met in this way is that it is impossible to get a well-defined braid group representation taking a limit $u \to \mathrm{i}\infty$, or some other one, for the two-colour Boltzmann weights $W(u)$. Also the formal sufficient condition for the construction of link invariants from $W(u)$ that one can find in [14] is not satisfied. It means that the regime we considered does not correspond to a conformal point. But one cannot reject a possibility of gapless behaviour for the models we considered in other regimes, as it was pointed out in [6] for dimerized coupled spin chain.

Let us stress an almost unrestricted amount of possibilities in the construction of new integrable multicoloured TL models demonstrated above. Certainly, these zoo of new models deserve a classification which, as far as we know, is missing for today.

An interesting possible alternative is coordinate Bethe ansatz formulated in terms of the Fuss–Catalan algebra itself. Such a kind of Bethe ansatz formulation, purely in terms of usual TL algebra generators for the XXZ model, was applied in [15]. The basis for wavefunction representation was chosen as a basis of specially defined two-sided ideals of words of TL generators. It would be interesting to apply such a formulation to the multicoloured TL lattice models.

This paper is only a first touch to these new integrable models. Development and applications of known methods of integrable models, such as algebraical Bethe ansatz, transfer matrix functional relations leading to a (system of) integral equations, a role of Q-operator in these models and so on, are open and interesting problems.

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